International Journal of Theoretical Physics, Vol. 34, No. 4, 1995

Integrability Analysis of a Conformal Equation in Relativity

K. S. Govinder,[1] P. G. L. Leach,[1,2] and S. D. Maharaj[1]

Received July 5, 1994

In 1987 C. C. Dyer, G. C. McVittie, and L. M. Oattes derived the (two) field equations for shear-free, spherically symmetric perfect fluid spacetimes which admit a conformal symmetry. We use the techniques of the Lie and Painlevé analyses of differential equations to find solutions of these equations. The concept of a pseudo-partial Painlevé property is introduced for the first time which could assist in finding solutions to equations that do not possess the Painlevé property. The pseudo-partial Painlevé property throws light on the distinction between the classes of solutions found independently by P. Havas and M. Wyman. We find a solution for all values of a particular parameter for the first field equation and link it to the solution of the second equation. We indicate why we believe that the first field equation cannot be solved in general. Both techniques produce similar results and demonstrate the close relationship between the Lie and Painlevé analyses. We also show that both of the field equations of Dyer *et al.* may be reduced to the same Emden–Fowler equation of index two.

1. INTRODUCTION

If a system of ordinary differential equations possesses the Painlevé property, it is conjectured that it represents a completely integrable nonchaotic dynamical system. [See, for example, Ablowitz *et al.* (1980) and Bountis *et al.* (1982), among others. For a detailed introduction to the Painlevé test we refer the reader to the excellent review by Ramani *et al.* (1989).] It may happen that some systems only possess an incomplete number of constants that arise at the resonances. Systems exhibiting this phenomenon are said to possess the partial Painlevé property. Such systems can be interpreted as

[1] Department of Mathematics and Applied Mathematics, University of Natal, Dalbridge 4014, South Africa.
[2] Member of the Centre for Theoretical and Computational Chemistry, University of Natal, and Associate Member of the Centre for Nonlinear Studies, University of Witwatersrand, Johannesburg, South Africa.

being integrable on a p-dimensional submanifold in the q-dimensional phase space, where p is the number of constants found and q the degrees of freedom of the system ($p < q$). We believe that the partial Painlevé property was first demonstrated by Cotsakis and Leach (1994) in their analysis of the gravitational field of the Mixmaster Universe. Our intention in this paper is to demonstrate that, in addition to the Painlevé property and the partial Painlevé property, some systems exhibit a further identifiable property which we call the pseudo-partial Painlevé property. This concept will be fully described in Section 3.

In this paper we analyze a nonlinear third-order differential equation and a nonlinear second-order differential equation that arise in general relativity. These equations arise in the study of spherically symmetric gravitational fields that possess a conformal symmetry in the t–r plane (Dyer *et al.*, 1987). The third-order equation possesses the pseudo-partial Painlevé property and we can integrate it once to obtain a second-order nonlinear equation. For a particular value of the first integral this equation has the Painlevé property, has two symmetries, and can be reduced to quadratures. It is remarkable that an equation that possesses the pseudo-partial Painlevé property can be integrated to one that possesses the Painlevé property. This demonstrates that the pseudo-partial Painlevé property is important in the solution of differential equations that arise in practical applications. The second-order equation is analyzed using both the Lie and Painlevé methods. We show how this gives further evidence of the close relationship between these two methods of solving differential equations. We finally note that the problem of solving the two field equations of Dyer *et al.* (1987) is essentially that of solving a single Emden–Fowler equation of index two.

In the next section we introduce the field equations and indicate what solutions have already been found. In Section 3 we embark on a Painlevé analysis of the third-order equation and one of its first integrals. We show that this approach is parallel to that of using the Lie analysis by treating the first integral as an Emden–Fowler equation for a particular value of the first integral in Section 4. We finally analyze the second-order equation using the Lie and Painlevé analyses.

2. THE EQUATIONS OF DYER, MCVITTIE, AND OATTES

Spherically symmetric gravitational fields are important in relativistic astrophysics and cosmology (Kramer *et al.*, 1980). Such gravitational fields with vanishing shear and admitting a conformal Killing vector in the t–r plane have been investigated by Dyer *et al.* (1987). They generated the third-order field equation

$$\mu^2 T_{\mu\mu\mu} + \mu(2m - 1)T_{\mu\mu} + (m^2 - 2m + 2T)T_\mu = 0 \tag{2.1}$$

where T is related to the gravitational potential, $\mu = r/t$ is a self-similarity variable, and m is a constant. [For more information on conformal symmetries and their relationship to mathematical physics the reader is referred to Choquet-Bruhat *et al.* (1977).] Solutions to (2.1) are important because they help to generate solutions to the Einstein field equations. Dyer *et al.* (1987) did not present any solutions to (2.1). It was only recently that Maharaj *et al.* (1991) found solutions to (2.1) in the form of Weierstrass elliptic functions. Their solution has the form

$$y = a^2 C_1^2 e^{-2ax} \mathcal{P}(C_1 e^{-ax} + C_2, 0, -1) \tag{2.2}$$

where

$$\begin{aligned} T(\mu) &= \gamma y(x) + T_0, & x &= \ln(\mu)/\beta \\ 2(m-2)\beta &= 5a, & \bar{k}\beta^2 &= 6a^2 \\ \beta^2\gamma &= -6, & \bar{k} &= \pm[(m-1)^2(m-3)^2 + 4k]^{1/2} \end{aligned} \tag{2.3}$$

$\mathcal{P}$ is the Weierstrass elliptic function, k is the value of a first integral of (2.1), and C_1, C_2 are arbitrary constants. Note that the assumption implicit in (2.2) is that $m \neq 2$. For the case $m = 2$ a solution can be found by simple integration. This is given by (Havas, 1992)

$$y = \frac{6\lambda^2}{-\beta^2\gamma} \mathcal{P}\left(\lambda x + \lambda\left(-\frac{6}{\beta^2\gamma}\right)^{1/2}, \lambda^{-4}\left(\frac{\beta^2\gamma}{6}\right)^2 \frac{3\bar{k}^2}{\gamma^2}, \lambda^{-6}\left(-\frac{\beta^2\gamma}{6}\right)^3 b\right) - \frac{\bar{k}}{2\gamma} \tag{2.4}$$

where Havas used the homogeneity property of the Weierstrass function and its evenness in x, (2.3), $b = (\bar{k}/\gamma)^3 + 12s/(\beta^2\gamma)$, and s is a constant.

It is interesting to note that other well-known solutions of the Einstein field equations may be related to (2.1), as was observed by Havas. For example, we regain some of the Wyman solutions for particular values of m and the first integral of (2.1). These are given by

$$m = -3, \qquad T = -24\mu^4 \mathcal{P}(\mu^2 + a, 0, b) \tag{2.5a}$$

$$m = 7, \qquad T = -24\mu^{-4} \mathcal{P}(\mu^{-2} + a, 0, b) \tag{2.5b}$$

$$m = \frac{9}{7}, \qquad T = -\frac{24}{49}[\mu^{4/7} \mathcal{P}(\mu^{2/7} + a, 0, b) + 1] \tag{2.5c}$$

$$m = \frac{19}{7}, \qquad T = -\frac{24}{49}[\mu^{-4/7} \mathcal{P}(\mu^{-2/7} + a, 0, b) + 1] \tag{2.5d}$$

We have independently verified that the functions (2.5) are solutions of the field equation (2.1).

3. PAINLEVÉ ANALYSIS

Using the transformation

$$\mu = e^{x/(2m-4)}, \qquad T(\mu) = 2(m-2)^2 y(x) - \tfrac{1}{2}(m^2 - 4m + 3)$$

we rewrite (2.1) in the autonomous form

$$y''' + y'' + yy' = 0 \tag{3.1}$$

We proceed with the Painlevé analysis in the standard manner (Ablowitz *et al.*, 1980). First, setting

$$y = \alpha\chi^p$$

where

$$\chi = x - x_0$$

we find that the pole is of second order and $\alpha = -12$. Now setting

$$y = -12\chi^{-2} + \beta\chi^{r-2}$$

we obtain the resonances $r = -1$, 4, and 6. To verify that (3.1) passes the Painlevé test, we substitute the truncated Laurent expansion

$$y = -12\chi^{-2} + a_{-1}\chi^{-1} + a_0 + a_1\chi + a_2\chi^2 + a_3\chi^3 + a_4\chi^4 \tag{3.2}$$

into (3.1) and solve for the a_i $(i = -1, \ldots, 4)$. This procedure results in

$$a_{-1} = \frac{1}{125}$$

$$a_0 = \frac{1}{25}$$

$$a_1 = \frac{12}{5}$$

$$a_2 = -\frac{1}{12{,}500}$$

$$a_3 = -\frac{1}{187{,}500}$$

a_4 is arbitrary

As there are only two arbitrary constants as opposed to the required three corresponding to the three degrees of freedom for (3.1), this suggests that (3.1) possesses the partial Painlevé property of Cotsakis and Leach (1994). However, if we truncate the expansion at the first resonance, viz. make the substitution

$$y = -12\chi^{-2} + \cdots + a_2\chi^2$$

we obtain, after substitution into (3.1),

$$a_{-1} = \frac{1}{125}$$

$$a_0 = \frac{1}{25}$$

$$a_1 = \frac{12}{5}$$

$$a_2 \text{ is arbitrary}$$

The constant at the first resonance is initially arbitrary, but is restricted to a particular value when the arbitrary constant at the second resonance is introduced. We call this property the pseudo-partial Painlevé property. The solution to (3.1) is then

$$\begin{aligned} y = {} & \frac{-12}{(x - x_0)^2} + \frac{1}{125}\frac{1}{(x - x_0)} + \frac{1}{25} + \frac{12}{5}(x - x_0) - \frac{1}{12{,}500}(x - x_0)^2 \\ & - \frac{1}{187{,}500}(x - x_0)^3 + a_4(x - x_0)^4 + \cdots \end{aligned}$$

and we say that (3.1) is integrable on a surface in three-dimensional parameter space.

Our analysis would normally end at this point. However, we note that (3.1) can be easily integrated to obtain the first integral

$$y'' + y' + \tfrac{1}{2}y^2 = K \tag{3.3}$$

where K is a constant of integration and thus a parameter. In general, (3.3) does not possess the Painlevé property (or any degree thereof). Reduction via the only symmetry (Head, 1993), $\partial/\partial x$, results in an Abel equation of the second kind the solution of which, unsurprisingly, is not obvious. However, for $K = 18/625$, (3.3) *does* possess the Painlevé property and has the solution

$$y = \frac{-12}{(x - x_0)^2} + \frac{12}{5}\frac{1}{(x - x_0)} + \frac{1}{25} + \frac{1}{125}(x - x_0) - \frac{1}{12{,}500}(x - x_0)^2 - \frac{1}{187{,}500}(x - x_0)^3 + a_4(x - x_0)^4 + \frac{1 - 7{,}500{,}000}{9{,}375{,}000}a_4(x - x_0)^5 + \cdots \tag{3.4}$$

To make the solution of (3.3) more transparent, we use the transformation

$$Y = e^{x/5}\left(\frac{y}{5} - \frac{2}{25}\right), \qquad X = e^{-x/5}$$

This transformation is suggested by the fact that, when $K = 18/625$, (3.3) has the two symmetries

$$G_1 = \frac{\partial}{\partial x} \tag{3.5a}$$

$$G_2 = e^{x/5}\frac{\partial}{\partial x} + e^{x/5}\left(-\frac{2}{5}y + \frac{12}{125}\right)\frac{\partial}{\partial y} \tag{3.5b}$$

where we call (3.5b) a 'conditional' symmetry (see also Sarlet *et al.*, 1985). We can now rewrite (3.3) as

$$Y'' + 25Y^2 = 0 \tag{3.6}$$

which has the symmetries

$$G_1 = \frac{\partial}{\partial X} \tag{3.7a}$$

$$G_2 = X\frac{\partial}{\partial X} - 2Y\frac{\partial}{\partial Y} \tag{3.7b}$$

Since the transformation is nonsingular, (3.6) still has the Painlevé property and we write its solution as

$$\begin{aligned} Y(\chi) &= -\frac{6}{25}\chi^{-2} + a_4\chi^4 - \frac{25}{78}a_4^2\chi^{10} + \frac{625}{8892}a_4^3\chi^{16} - \frac{3125}{231{,}192}a_4^4\chi^{22} + \cdots \\ &= -\frac{6}{25}\left(\chi^{-2} + b_4\chi^4 - \frac{b_4^2}{13}\chi^{10} + \frac{b_4^3}{247}\chi^{16} - \frac{3b_4^4}{16{,}055}\chi^{22} + \cdots\right) \\ &= -\frac{6}{25}\wp(\chi) \end{aligned} \tag{3.8}$$

where $\chi = X - X_0$ and $\mathcal{P}(\chi)$ is the Weierstrass $\mathcal{P}$ function with $c_2 = 0$ and $c_3 = b_4 = (25/6)a_4$ (Abramowitz and Stegun, 1972). The form of $Y(\chi)$ is not surprising, as (3.6) is essentially the defining differential equation for the Weierstrass $\mathcal{P}$ function. Note that (3.8) is the solution to (3.3) [and hence (3.1)] for a particular value of the first integral K, but for all values of m, in contrast to Wyman's solutions (2.5), which hold only for the particular values of m.

4. EMDEN–FOWLER APPROACH

We can transform (3.3) into the standard form of the Emden–Fowler equation. Setting

$$y = 2z(x) + b$$

where $K = \frac{1}{2}b^2$, we write (3.3) as

$$z'' + z' + bz + z^2 = 0 \tag{4.1}$$

We remove the z' and z terms in (3.3) using the well-known Kummer–Liouville transformation (Kummer, 1887; Liouville, 1837)

$$z(x) = u(x)v(t), \qquad t = t(x)$$

where, in our case,

$$u(x) = e^{-(1+\alpha)x/2}$$

$$t(x) = \frac{1}{|\alpha|}\, e^{\alpha x}$$

$$\alpha = (1 - 4b)^{1/2}$$

Equation (4.1) then becomes

$$\ddot{v} + (|\alpha| t)^{-(1+5\alpha)/(2\alpha)} v^2 = 0 \tag{4.2}$$

For our particular value of K, 18/625, α has the four values $\pm 1/5$, $\pm 7/5$. This gives the set of equations

$$\ddot{v} + v^2 = 0, \qquad \alpha = -\frac{1}{5} \tag{4.3a}$$

$$\ddot{v} + \left(\frac{t}{5}\right)^{-5} v^2 = 0, \qquad \alpha = \frac{1}{5} \tag{4.3b}$$

$$\ddot{v} + \left(\frac{t}{5}\right)^{-15/7} v^2 = 0, \qquad \alpha = -\frac{7}{5} \tag{4.3c}$$

$$\ddot{v} + \left(\frac{t}{5}\right)^{-20/7} v^2 = 0, \qquad \alpha = \frac{7}{5} \tag{4.3d}$$

It is easy to eliminate the constant coefficients of v^2 and consider the transformed system

$$\ddot{v} + v^2 = 0 \tag{4.4a}$$

$$\ddot{v} + t^{-5}v^2 = 0 \tag{4.4b}$$

$$\ddot{v} + t^{-15/7}v^2 = 0 \tag{4.4c}$$

$$\ddot{v} + t^{-20/7}v^2 = 0 \tag{4.4d}$$

with the corresponding symmetries

$$G_1 = \frac{\partial}{\partial t}, \qquad G_2 = t\frac{\partial}{\partial t} - 2v\frac{\partial}{\partial v} \tag{4.5a}$$

$$X_1 = t^2\frac{\partial}{\partial t} + vt\frac{\partial}{\partial v} \qquad X_2 = -t\frac{\partial}{\partial t} - 3v\frac{\partial}{\partial v} \tag{4.5b}$$

$$Y_1 = 343t^{6/7}\frac{\partial}{\partial t} + (147t^{-1/7}v - 12)\frac{\partial}{\partial v} \qquad Y_2 = 7t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v} \tag{4.5c}$$

$$Z_1 = 343t^{8/7}\frac{\partial}{\partial t} + (196t^{1/7}v + 125)\frac{\partial}{\partial v} \qquad Z_2 = -7t\frac{\partial}{\partial t} - 6v\frac{\partial}{\partial v} \tag{4.5d}$$

Since the Lie bracket of each pair is

$$[G_1, G_2] = G_1$$

we can reduce each equation in (4.4) to quadratures (Olver, 1993) using G_1. Equation (4.4a) can be reduced to the elliptic integral

$$t - t_0 = \int \frac{dv}{[C_0 - (2v^3)/3]^{1/2}} \tag{4.6}$$

or regarded as

$$v = \mathcal{P}((-1/6t)^{1/2} + a, 0, b) \tag{4.7}$$

where $\mathcal{P}$ is again the Weierstrass $\mathcal{P}$ function (Kamke, 1983; Gradshteyn and Ryzhik, 1980) and a and b are arbitrary constants.

We can relate the solutions of (4.4b) to (4.7) by the mapping

$$t \to \frac{25}{t} \tag{4.8a}$$

$$v \to \frac{5v}{t} \tag{4.8b}$$

of (4.4c) by

$$t \to (49t)^7 \tag{4.9a}$$

$$v \to vt^3 + 6t \tag{4.9b}$$

and of (4.4d) by

$$t \to -\frac{1}{(49t)^7} \tag{4.10a}$$

$$v \to \frac{1}{49^7 t^4}\left(v + \frac{6}{t^2}\right) \tag{4.10b}$$

Note that Leach *et al.* (1992) showed that (4.4d) can be reduced to

$$7(t_0^{-1/7} - t^{-1/7}) = \int_{p_0}^{p} \frac{d\eta}{[2I - (2\eta^3)/3]^{1/2}} \tag{4.11}$$

where p was one of the two integral invariants of Z_2, i.e., one of

$$p = t^{-4/7}v - \frac{6}{49}t^{2/7} \tag{4.12a}$$

$$q = t^{4/7}v - \frac{4}{7}t^{-3/7} - \frac{12}{343}t^{3/7} \tag{4.12b}$$

and also related (4.4c) to (4.4d) using the mapping

$$t \to t^{-1} \tag{4.13a}$$

$$v \to \frac{v}{t} \tag{4.13b}$$

5. ANALYSIS OF THE SECOND FIELD EQUATION

Havas (1992) showed that the second field equation of Dyer *et al.* (1987) could be written in the form

$$w'' - 6x^{-5n}w^2 = 0 \tag{5.1}$$

Following the idea of Mellin *et al.* (1994), we analyze (5.1) by requiring that it possess two point symmetries and hence be integrable.

Recall that a system of second-order ordinary differential equations

$$\mathbf{N}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}, t) = 0$$

possesses a symmetry of the form

$$G = \xi(\mathbf{x}, t) \frac{\partial}{\partial t} + \eta_i(\mathbf{x}, t) \frac{\partial}{\partial x_i}$$

if

$$G^{[2]}\mathbf{N}|_{\mathbf{N}=0} = 0$$

where

$$G^{[2]} = G + (\dot{\eta}_i - \dot{x}_i\dot{\xi}) \frac{\partial}{\partial \dot{x}_i} + (\ddot{\eta}_i - 2\ddot{x}_i\dot{\xi} - \dot{x}_i\ddot{\xi}) \frac{\partial}{\partial \ddot{x}_i}$$

is the second extension of G. (Note that the overdot represents total differentiation with respect to time.)

We require that (5.1) be invariant under a symmetry of the form (Mellin *et al.*, 1994)

$$G = a(x) \frac{\partial}{\partial x} + [b(x)w + c(x)] \frac{\partial}{\partial w} \tag{5.2}$$

Separation by powers of w' and w results in the following system of ordinary differential equations:

$$2b' = a'' \tag{5.3a}$$

$$6(b - 2a')x^{-5n} = -30anx^{-5n-1} + 12bx^{-5n} \tag{5.3b}$$

$$b'' = 12cx^{-5n} \tag{5.3c}$$

$$c'' = 0 \tag{5.3d}$$

Equations (5.3d) and (5.3c) give c and hence b as

$$c = C_0 + C_1x \tag{5.4}$$

$$b = B_0 + B_1x + \frac{12C_0x^{-5n+2}}{(-5n + 1)(-5n + 2)} + \frac{12C_1x^{-5n+3}}{(-5n + 2)(-5n + 3)} \tag{5.5}$$

In general, we can write a, given by (5.3b) as

$$a = A_1x^{5n/2} - \frac{B_0x}{-5n + 2} - \frac{B_1x^2}{-5n + 4} - \frac{6C_0x^{-5n+3}}{(-5n + 1)(-5n + 2)(-15n/2 + 3)}$$
$$- \frac{6C_1x^{-5n+4}}{(-5 + 2)(-5n + 3)(-15n/2 + 4)} \tag{5.6}$$

The final step in determining the form of (5.2) is to satisfy the consistency condition given by (5.3a), which we can now write as

$$
\begin{aligned}
2B_1 &+ \frac{24C_0x^{-5n+1}}{(-5n+1)} + \frac{24C_1x^{-5n+2}}{(-5n+2)} \\
&= \frac{5n}{2}\left(\frac{5n}{2} - 1\right)A_1x^{5n-2} - \frac{B_1}{-5n/2+2} \\
&\quad - \frac{6(-5n+3)C_0x^{-5n+1}}{(-5n+1)(-15n/2+3)} - \frac{6(-5n+4)C_1x^{-5n+2}}{(-5n+2)(-15n/2+4)}
\end{aligned} \tag{5.7}
$$

Note that B_0 does not appear in (5.7), implying that we always have at least one symmetry which has the form

$$G_1 = x\frac{\partial}{\partial x} + (5n-2)w\frac{\partial}{\partial w} \tag{5.8}$$

In addition, we have a possible second symmetry by equating coefficients of the powers of x in (5.7) to zero. Thereafter, setting each of the constants B_1, C_0, C_1 except one (which is set equal to one) in turn to zero, we obtain appropriate values for n, i.e., for B_1, $n = 1$; C_0, $n = 3/7$; and for C_1, $n = 4/7$. Further for A_1 we have $n = 0, 2/5$. However, $n = 2/5$ makes the coefficient of C_1 infinite and is therefore invalid. These values of n imply that the equations

$$w'' - 6w^2 = 0, \qquad n = 0 \tag{5.9a}$$

$$w'' - 6x^{-15/7}w^2 = 0 \qquad n = \frac{3}{7} \tag{5.9b}$$

$$w'' - 6x^{-20/7}w^2 = 0 \qquad n = \frac{4}{7} \tag{5.9c}$$

$$w'' - 6x^{-5}w^2 = 0 \qquad n = 1 \tag{5.9d}$$

have the corresponding pairs of symmetries

$$G_1 = -x\frac{\partial}{\partial x} + 2w\frac{\partial}{\partial w}, \qquad G_2 = \frac{\partial}{\partial x} \tag{5.10}$$

$$X_1 = 7x\frac{\partial}{\partial x} + w\frac{\partial}{\partial w}, \qquad X_2 = 343x^{6/7}\frac{\partial}{\partial x} + (147x^{-1/7}w + 2)\frac{\partial}{\partial w} \tag{5.11}$$

$$Y_1 = 7x\frac{\partial}{\partial x} + 6w\frac{\partial}{\partial w}, \qquad Y_2 = 343x^{8/7}\frac{\partial}{\partial x} + (196x^{1/7}w - 2)\frac{\partial}{\partial w} \tag{5.12}$$

$$Z_1 = x\frac{\partial}{\partial x} + 3w\frac{\partial}{\partial w}, \qquad Z_2 = x^2\frac{\partial}{\partial x} + xw\frac{\partial}{\partial w} \tag{5.13}$$

The constant coefficients of the nonlinear terms in (5.9) can be transformed away. This reduces (5.9) to the system (4.4) and the discussion of following (4.4) applies equally in this case. Thus we have reduced the problem of solving the two field equations of Dyer *et al.* (1987) to that of solving the single equation

$$y'' + x^n y^2 = 0 \tag{5.14}$$

Mellin *et al.* showed that the equation

$$y'' + p(x)y' + r(x)y = f(x)y^n \tag{5.15}$$

can be transformed to the autonomous form

$$Y'' + 2C_0Y' + (M + C_0^2)Y + N = KY^2 \tag{5.16}$$

where C_0 is given by our B_0 in (5.5),

$$M = \frac{1}{2}aa'' - \frac{1}{4}a'^2 - \frac{1}{2}\left(p' + \frac{1}{2}p^2 - 2r\right)a^2$$
$$- 2K\int\frac{d}{a^{3/2}}\exp\left[\frac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right] \tag{5.17}$$

and

$$N = K\left\{\int\frac{d}{a^{3/2}}\exp\left[\frac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right]\right\}^2 - \int\left\{\left[\frac{1}{2}aa'''\right.\right.$$
$$- \left(p' + \frac{1}{2}p^2 - 2r\right)aa'$$
$$\left.\left.- \frac{1}{2}\left(p' + \frac{1}{2}p^2 - 2r\right)a^2\right]\left[\int\frac{d}{a^{3/2}}\exp\left[\frac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right]\right]\right\} \tag{5.18}$$

where a is given in (5.2), K is a constant of integration, and d is our c in (5.2), provided $f(x)$ is given by

$$f(x) = Ka^{-5/2}\exp\left[\frac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right] \tag{5.19}$$

Note that (5.17) and (5.18) imply $N = 0$. Thus (5.16) is more correctly written as

$$Y'' + 2C_0Y' + (M + C_0^2)Y = KY^2 \tag{5.20}$$

The transformation that takes (5.15) to (5.20) is

$$X = \int \frac{dx}{a} \tag{5.21a}$$

$$Y = y \exp\left(-\int \frac{c}{a}\right) - \int \left[\frac{d}{a} \exp\left(-\int \frac{c}{a}\right)\right] \tag{5.21b}$$

where c is our b in (5.2) and is given by

$$c = C_0 + \tfrac{1}{2}(a' - ap) \tag{5.22}$$

Mellin *et al.* (1994) showed that (5.16) had two symmetries iff (now taking $N = 0$ into account)

$$\left(M + \frac{C_0^2}{25}\right)\left(M + \frac{49C_0^2}{25}\right) = 0 \tag{5.23}$$

Our analysis is much simpler, as we do not have the functions $p(x)$ and $r(x)$, a is a quadratic in x, and $f(x)$ is explicitly given by

$$f(x) = x^{-5n} \tag{5.24}$$

The condition (5.23) restricts n in (5.24), via (5.17), to 0, 3/7, 4/7, and 1/5. We note that (5.20) is also integrable when $C_0 = 0$. This gives $n = 1/2$ and (5.1) becomes

$$w'' - 6x^{-5/2}w^2 = 0 \tag{5.25}$$

Equations (5.9) and (5.25) are exactly those for which Wyman (1976) found solutions, as given by Havas (1992).

Noting the parallels between the two field equations, one is tempted to find an analogy for $C_0 = 0$ in the case of the first equation. However, this is not possible, as the coefficient of z' in (4.1) is fixed.

It is interesting to analyze our version of (5.20), viz.

$$Y'' + 5(2n - 1)Y' + (5n - 2)(5n - 3)Y = 6Y^2 \tag{5.26}$$

using the Painlevé method to determine if further solutions can be found. The pole is at -2 and the resonances occur at $r = -1, 6$. Upon substituting the Laurent expansion

$$Y = \sum_{i=-2}^{6} a_i \chi^i \tag{5.27}$$

where

$$\chi = X - X_0$$

into (5.26), we find that it has the Painlevé property only if $n = 0$, 1/5, 3/7, 1/2, or 4/7. This is equivalent to (5.23) or $C_0 = 0$ for (5.20). While not providing new solutions, this is further evidence of the close relationship between the Lie and Painlevé analyses of differential equations.

6. CONCLUSION

Recently Mellin *et al.* (1994) treated cases of the generalized Emden–Fowler equation that have two symmetries. For $n = 2$ the condition they require for the equation to have two symmetries is exactly that our K in (3.3) must equal 18/625, a value we obtained by requiring that (3.3) possess the Painlevé property. This is another example of the close link between the Painlevé and Lie analyses of ordinary differential equations. It further demonstrates the usefulness of the Painlevé property in determining the integrability of differential equations. This strongly suggests that the interrelationship between the Painlevé and Lie analyses should be thoroughly investigated. This project is currently under investigation.

The fact that the original equation (3.1) possessed the pseudo-partial Painlevé property suggests that this phenomenon could be used to find solutions to equations that do not have the Painlevé property. The Einstein field equations are extremely nonlinear and it is difficult to find exact solutions even in the special case of spherical symmetry. The pseudo-partial Painlevé property provides a systematic approach in the search for new solutions of physical interest. This approach should be applied to other useful differential equations arising in cosmology and relativistic astrophysics in spacetimes not necessarily containing spherical symmetry.

We emphasize that the possession of the pseudo-partial Painlevé property merely suggests that the equation is integrable on a subspace of the space of initial conditions (Cotsakis and Leach, 1994). We cannot predict the behavior of the solution off the subspace. In fact, it has recently been shown that a third-order system that possesses the partial Painlevé property on a known subspace is chaotic in the sense of Lyapunov for sets of initial conditions off that surface (Richard and Leach, 1994). This is supported by the fact that we are only able to find solutions to the first field equation for a particular value of its first integral, i.e., this value determines the subspace on which the equation is integrable. This is in agreement with the ARS conjecture (Ablowitz *et al.*, 1981), which holds only for this value of the first integral. Thus we expect solutions of the first field equation to be found, at best, for particular values of the first integral and all m or vice versa. This explains Havas' results (Havas, 1992), which are only valid for particular values of the first integral, and Wyman's results (Wyman, 1976), which only hold for particular values of m.

We finally note that the search for further solutions of the Dyer–McVittie–Oattes field equations (Dyer *et al.*, 1987) should be confined to finding the different values of n for which

$$y'' + x^n y^2 = 0$$

is integrable.

ACKNOWLEDGMENT

The Foundation for Research and Development of South Africa is thanked for its continuing support.

REFERENCES

Ablowitz, M. J., Ramani, A., and Segur, H. (1980). *Journal of Mathematical Physics*, **21**, 715.

Abramowitz, M., and Stegun, I. A. (1972). *Handbook of Mathematical Functions*, Dover, New York, Chapter 18.

Bountis, T., Segur, H., and Vivaldi, F. (1982). *Physical Review A*, **25**, 1257.

Choquet-Bruhat, Y., Dewitt-Morrette, C., and Dillard-Bleick, M. (1977). *Analysis, Manifolds and Physics*, North-Holland, Amsterdam.

Cotsakis, S., and Leach, P. G. L. (1994). *Journal of Physics A: Mathematics and General*, **27**, 1625.

Dyer, C. C., McVittie, G. C., and Oattes, L. M. (1987). *General Relativity and Gravitation*, **19**, 887.

Gradshteyn, I. S., and Ryzhik, I. M. (1980). *Table of Integrals, Series and Products*, Academic Press, New York, p. 631.

Havas, P. (1992). *General Relativity and Gravitation*, **24**, 599.

Head, A. K. (1993). *Computer Physics Communications*, **77**, 241.

Kamke, E. (1983). *Differentialgleichungen Lösungsmethoden und lösungen I*, Teubner, Stuttgart, p. 542.

Kramer, D., Stephani, H., MacCallum, M. A. H., and Herlt, E. (1980). *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, Cambridge, Chapter 14.

Kummer, E. (1887). *Journal Reine Angewandte Mathematik*, **100**, 1.

Leach, P. G. L., Maartens, R., and Maharaj, S. D. (1992). *International Journal of Nonlinear Mechanics*, **27**, 575.

Liouville, J. (1837). *Journal de Mathematiques Pures et Appliques*, **2**, 16.

Maharaj, S. D., Leach, P. G. L., and Maartens, R. (1991). *General Relativity and Gravitation*, **23**, 261.

Mellin, C. M., Mahomed, F. M., and Leach, P. G. L. (1994). *International Journal of Nonlinear Mechanics*, to appear.

Olver, P. J. (1993). *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, p. 149.

Ramani, A., Grammaticos, B., and Bountis, T. (1989). *Physics Reports*, **180**, 160.

Richard, A. C., and Leach, P. G. L. (1994). *The integrability of* $y''' + y'' + yy' = 0$, *Journal of Physics A: Mathematical and General*, submitted.

Sarlet, W., Leach, P. G. L., and Cantrijn, F. (1985). *Physica*, **17D**, 87.

Wyman, M. (1976). *Canadian Mathematical Bulletin*, **19**, 343.